Math 3063 Abstract Algebra — Prof. Paul Bailey — Project 3 — March 2, 2009 — Name:

Due Friday, February 20, 2009.

Copy the statement of the problem on a piece of $8\frac{1}{2} \times 11$ piece of blank computer paper, and write the solution underneath. Write neatly. Mathematics should always be written in grammatically correct English, in complete sentences.

Definition 1. Let G be a group. We say that G is *abelian* if $gh = hg$ for all $g, h \in G$.

Problem 1. Let G be a group such that $g^2 = 1$ for every $g \in G$. Show that G is abelian.

Definition 2. Let G be a group and let $g \in G$. The *order* of G is $|G|$. The *order* of g is the smallest positive integer k such that $g^k = 1$.

Problem 2. Let G be a group of even order. Show that G has an element of order two.

Definition 3. Let G be a group and let $H \subset G$. We say that H is a *subgroup* of G, and write $H \leq G$, if

(S0) H is nonempty;

(S1) $h_1, h_2 \in H \Rightarrow h_1 h_2 \in H$;

(S2) $h \in H \Rightarrow h^{-1} \in H$.

Problem 3. Let $V = \mathbb{R}^3$; this is a group under addition. Let $T : V \to \mathbb{R}$ be given by $T(x, y, z) = x + y + z$. Let $W = \{\vec{v} \in \mathbb{R}^3 \mid T(\vec{v}) = 0\}$. Show that $W \leq V$.

Definition 4. Let G be a group. The *center* of G is

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}.$$

Problem 4. Let G be a group with a unique element $g \in G$ of order two. Show that $g \in Z(G)$.

Definition 5. Let G be a group and let $H \leq G$.

The *centralizer of H in G* is

$$C_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

Problem 5. Let G be a group, $g \in G$, and $H \leq G$. The *centralizer* of g in H is

$$C_H(g) = \{h \in H \mid h^{-1} g h = g\}.$$

Show that $C_G(g) \leq G$.

Definition 6. Let G be a group and let $g, h \in G$. The *conjugate* of h by g is $g^{-1} h g$.

Problem 6. Let G be a group and let $g, h \in G$.

(a) Show that g and h commute if and only if $g^{-1} h g = h$.

(b) Show that $(g^{-1} h g)^n = g^{-1} h^n g$ (use induction).

(c) Show that if $\operatorname{ord}(h) = n$, then $\operatorname{ord}(g^{-1} h g) = n$.